$$\int_{\partial S = C} \vec{F}\cdot d\vec{r} = \iint_S \operatorname{rot}\vec{F}\cdot d\vec{S}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int_C \vec{F}\cdot\vec{T}\, ds = \iint_S \operatorname{rot}\vec{F}\cdot\vec{n}\, dS$$

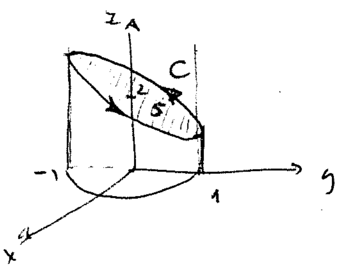
integral de línea de la componente tangencial de $\vec{F}$ alrededor de la curva frontera de S. = integral de superficie de la componente normal del $\operatorname{rot}\vec{F}$

recordar que $\vec{T} = \dfrac{\frac{d\vec{r}}{dt}}{\left\| \frac{d\vec{r}}{dt} \right\|}$

por otro lado $\dfrac{ds}{dt} = \left\| \dfrac{d\vec{r}}{dt} \right\|$ y $\dfrac{d\vec{r}}{ds} = \dfrac{d\vec{r}}{dt} \cdot \dfrac{dt}{ds} = \dfrac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \dfrac{\frac{d\vec{r}}{dt}}{\left\|\frac{d\vec{r}}{dt}\right\|} = \vec{T}$

por lo tanto $\dfrac{d\vec{r}}{ds} = \vec{T} \longrightarrow d\vec{r} = \vec{T}\cdot ds$

Ejemplo: Evalue $\displaystyle\int_{\partial S\, C} \vec{F}\cdot d\vec{r}$, donde $\vec{F}(x,y,z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ y C es la curva de intersección del plano $y+z=2$ con el cilindro $x^2+y^2=1$. (Orientada C de manera que se recorra en sentido contrario al de las manecillas del reloj)



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{rot}\vec{F}\cdot\vec{n}\, dS$$

$$\operatorname{rot}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (0-0,\ 0-0,\ +2y) = (0,\ 0,\ 1+2y)$$

paramétrica de C

$x = x$, $y = y$ $\quad z = g(x,y)$
$\quad z = 2-y$

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\, dA = \sqrt{2}\, dA.$$

$$\vec{n} = \frac{\left(-\frac{\partial g}{\partial x},\ \frac{\partial g}{\partial y},\ 1\right)}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} = \frac{(0,\ -1,\ 1)}{\sqrt{1+0+1}} = \frac{(0,-1,1)}{\sqrt{2}}$$

$$\int_C \vec{F}\, d\vec{r} = \iint_S \text{rot}\, \vec{F} \cdot \vec{n}\, dS.$$

$$= \iint_D \left(\nabla \times \vec{F}\right) \frac{(0,-1,1)}{\sqrt{2}} \cdot \sqrt{2}\; dA.$$

$$= \iint_D (0,0,1+2y)(0,-1,1)\; dA$$

$$= \iint_D (1+2y)\; dA$$

En este caso D es el disco de radio 1. y en coord. polares.

$$= \int_0^{2\pi}\int_0^1 (1 + 2r\,\text{sen}\,\theta)\; r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r + 2r^2\,\text{sen}\,\theta\right) dr\, d\theta$$

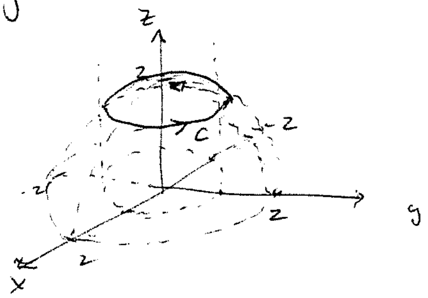
$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3}\,\text{sen}\,\theta\right]\Bigg|_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\,\text{sen}\,\theta\right) d\theta$$

$$= \frac{1}{2}\theta\Big|_0^{2\pi} - \frac{2}{3}\cos\theta\Big|_0^{2\pi}$$

$$= \frac{2\pi}{2} - \frac{2}{3}\Big(\underbrace{\cos 2\pi}_{1} - \underbrace{\cos 0}_{1}\Big) = \pi$$

Ejemplo: utilice el teorema de Stokes para calcular la integral $\iint_S \text{rot}\,\vec{F}\cdot d\vec{S}$ donde $\vec{F} = (yz, xz, xy)$ y S es la parte de la esfera $x^2+y^2+z^2=4$ que se encuentra dentro del cilindro $x^2+y^2=1$ y arriba del plano xy.



$$\left.\begin{array}{l} x^2+y^2+z^2=4 \\ x^2+y^2=1 \end{array}\right]$$

Para hallar la curva C

$$1+z^2=4$$
$$z^2=3 \longrightarrow z=\sqrt{3} \quad (+)$$
$$z>0$$

Así C es el círculo dado por $x^2+y^2=1$, $z=\sqrt{3}$

la ecuación vectorial de C es

$$\vec{r}(t) = \cos t\,\hat{\imath} + \operatorname{sen} t\,\hat{\jmath} + \sqrt{3}\,\hat{k} \qquad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = -\operatorname{sen} t\,\hat{\imath} + \cos t\,\hat{\jmath} + 0\,\hat{k}$$

$$\vec{F}(\vec{r}(t)) = \sqrt{3}\operatorname{sen} t\,\hat{\imath} + \sqrt{3}\cos t\,\hat{\jmath} + \cos t \operatorname{sen} t\,\hat{k}$$

por el teorema de Stokes

$$\iint_S \text{rot}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\operatorname{sen} t, \sqrt{3}\cos t, \cos t \operatorname{sen} t)\cdot(-\operatorname{sen} t, \cos t, 0)\,dt$$

$$= \int_0^{2\pi} (-\sqrt{3}\operatorname{sen}^2 t + \sqrt{3}\cos^2 t)\,dt = \sqrt{3}\int_0^{2\pi} (\cos^2 t - \operatorname{sen}^2 t)\,dt$$

$$= \sqrt{3}\int_0^{2\pi} \cos 2t\,dt = 0.$$

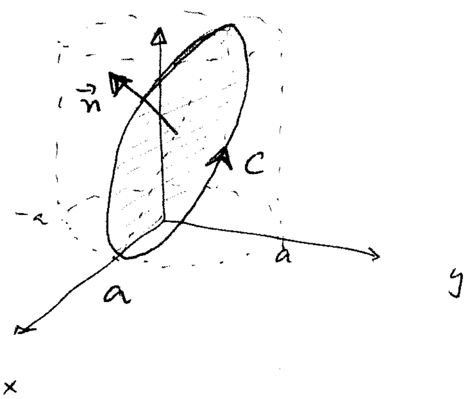
$$\cos 2\theta = \cos^2\theta - \operatorname{sen}^2\theta$$

Ejercicio:

Calcular $\oint_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$,

siendo C la curva de intersección del cilindro

$x^2+y^2=a^2$ y el plano $\frac{x}{a}+\frac{z}{b}=1$; $a,b>0$.



Cuando:

$x=a \longrightarrow z=0$

$x=0 \longrightarrow z=b$

$x=-a \longrightarrow z=2b$.

Aplicar Stokes.

~~Sea~~ Sea S la elipse que se produce al intersectar las dos superficies.

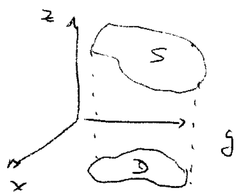
Aplicamos el teorema de Stokes.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{rot}\,\vec{F}\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot d\vec{s}$$

$$= \iint_S (\nabla\times\vec{F})\cdot\vec{n}\; ds$$

Nota:
$$\iint_S \vec{H}\cdot d\vec{s} = \iint_S \vec{H}\cdot\vec{n}\; ds = \iint_D (H_1, H_2, H_3)\cdot \overbrace{\left[\frac{\left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}}\right]}^{\vec{n}} \underbrace{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\; dx\,dy}_{ds}$$

- donde la superficie S, esta parametrizada por
  $x = x, \quad y = y, \quad z = g(x,y)$



- $$\vec{n} = \frac{-\frac{\partial g}{\partial x}\,\hat{i} - \frac{\partial g}{\partial y}\,\hat{j} + \hat{k}}{\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}}$$

- $\vec{H} = (H_1, H_2, H_3)$

- $ds = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\; \underbrace{dA}_{dx\,dy}$

finalmente

$$\iint_S \vec{H}\cdot d\vec{s} = \iint_D (H_1, H_2, H_3)\cdot\left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) dx\,dy$$

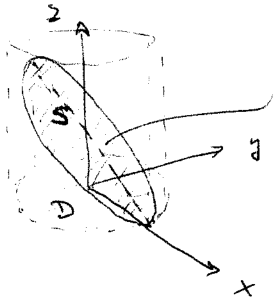
$$= \iint_D \left(-H_1\frac{\partial g}{\partial x} - H_2\frac{\partial g}{\partial y} + H_3\right) dx\,dy. \qquad \text{cartesianas}$$

$$=$$

$$\operatorname{rot}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{matrix} = \frac{\partial}{\partial y}(x-y)\,\hat{\imath} + \frac{\partial}{\partial x}(z-x)\,\hat{k} + \frac{\partial}{\partial z}(y-z)\,\hat{\jmath}$$

$$- \frac{\partial}{\partial y}(y-z)\,\hat{k} - \frac{\partial}{\partial z}(z-x)\,\hat{\imath} - \frac{\partial}{\partial x}(x-y)\,\hat{\jmath}$$

$$= -1\hat{\imath} - 1\hat{k} - 1\hat{\jmath} - 1\hat{k} - 1\hat{\imath} - 1\hat{\jmath} = (-2,-2,-2) = (H_1, H_2, H_3)$$



$$z = b - \frac{b}{a}x$$

S superficie

$$x = x\ ,\ y = y\ ,\ z = g(x,y) = b - \frac{b}{a}x$$

$$\frac{\partial g}{\partial x} = -\frac{b}{a}$$

$$\frac{\partial g}{\partial y} = 0$$

$$\vec{n} = -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1$$

$$ds = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dA$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\, ds = \iint_S \vec{H}\cdot\vec{n}\, ds$$

$$= \iint_D \left(-H_1\frac{\partial g}{\partial x} - H_2\frac{\partial g}{\partial y} + H_3\right) dx\, dy$$

$$= \iint_D \left(2\cdot\left(\frac{-b}{a}\right) + 2\cdot 0 - 2\right) dx\, dy = \iint_D \left(-\frac{2b}{a} - 2\right) dx\, dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{a} \left(-\frac{2b}{a} - 2\right) r\, dr\, d\theta = \int_0^{2\pi} \left(-\frac{2b}{a} - 2\right) \int_0^{2\pi} \frac{r^2}{2}\Big|_0^a d\theta$$

$$= \left(-\frac{2b}{a} - 2\right)\int_0^{2\pi} \frac{a^2}{2}\, d\theta = \left(-\frac{2b}{a} - 2\right)\frac{a^2}{2}\int_0^{2\pi} d\theta =$$

$$= -\frac{2b-2a}{a}\cdot\frac{a^2}{\cancel{2}}\cdot\cancel{2}\pi = -2(b+a)\cdot a\pi$$